

Linear Canonical Transformations and Quantum Phase: A unified canonical and algebraic approach

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Abstract

The algebra of generalized linear quantum canonical transformations is examined in the perspective of Schwinger's unitary-canonical operator basis. Formulation of the quantum phase problem within the theory of quantum canonical transformations and in particular with the generalized quantum action-angle phase space formalism is established and it is shown that the conceptual foundation of the quantum phase problem lies within the algebraic properties of the canonical transformations in the quantum phase space. The representations of the Wigner function in the generalized action-angle unitary operator pair for certain Hamiltonian systems with dynamical symmetry are examined. This generalized canonical formalism is applied to the quantum harmonic oscillator to examine the properties of the unitary quantum phase operator as well as the action-angle Wigner function.

I. INTRODUCTION AND REVIEW

The quantum mechanical operator realization of the classical phase observable, well-known as the historical quantum phase problem, is one of the oldest problems in quantum mechanics. In the quest for a correspondence between the classical action-angle variables and their quantum counterparts, Born, Heisenberg and Jordan have investigated¹ the problem in the earliest days of quantum mechanics in the general perspective of building a theory of quantum canonical transformations and their unitary representations. The search for a quantum phase operator within this canonical perspective has specifically begun in one of Dirac's early works² in 1927 where the principal motivation was to extend the principle of correspondence to that between the classical action-angle (AA) variables and their quantum counterparts. The quantum phase problem was then followed by the works of Heitler³ and Louisell⁴ where it was examined in terms of the quantization of the electromagnetic field. The introduction of *trigonometric Hermitian* phase operators by Susskind and Glogower⁵ created a trigonometric approach to the phase problem. At this point, a landmark was made by the introduction of the coherent state formalism by Glauber⁶ and, with the development of laser physics in the 1960s, the theoretical and experimental investigation of the properties of quantum phase became mainstream in quantum optics. On the other hand, contemporary to Glauber's work, Carruthers and Nieto in their seminal paper⁷, wisely entitled as *Phase and angle variables in quantum mechanics*, revived the interest on the canonical approach advocated in the early days of quantum mechanics. Since our interest in the current work is within the canonical perspective, we will refer the interested reader to, for instance, some recent reviews on the quantum phase as seen from the perspective of quantum optics.^{8,9} In search for the quantum counterpart of the classical AA pair, the canonical perspective in the quantum phase problem was furthered mainly by the works of Rocca and Siruge¹⁰, Boyer and Wolf¹¹, Moshinsky and Seligman¹², as well as Luis and Sanchez-Soto¹³ and more recently by Lewis et al.¹⁴ In an earlier work¹⁵, we introduced a different canonical-algebraic approach to the quantum phase problem from those in Ref's [11-14] by starting from the

generalized discrete unitary-cyclic finite (D) dimensional representations of the quantum phase space distribution functions in terms of Schwinger's operator basis.^{16,17} There are two crucial properties of these representations from the quantum phase operator point of view. The first one is that Schwinger's operator basis supports discrete cyclic finite dimensional subalgebraic representations with non-negative norms in the D dimensional Hilbert space H_D . These cyclic and admissible representations are known to be crucial for the existence of the phase operator in an arbitrary but finite dimensional algebra. On the other hand, the second crucial property is connected with the fact that the complete set of elements of the discrete finite dimensional cyclic Schwinger operator basis are the generalized dual representations of the standard Wigner-Kirkwood (WK)¹⁸ ones of the quantum phase space.¹⁵⁻¹⁷ Moreover, these elements are the generators of the discrete area preserving diffeomorphism on the two-dimensional toroidal lattice $\mathbb{Z}_D \times \mathbb{Z}_D$ which are known to respect the Fairlie-Fletcher-Zachos (FFZ) sine algebra.¹⁹ As the dimension D is extended to infinity, a limit to continuum can be realized where the connection with Arnold's infinitesimal area preserving diffeomorphism²⁰ on the continuous 2-torus is established. Hence, the representations of the quantum phase space in terms of Schwinger's unitary-canonical operator basis paves a direct route to the algebraic formulation of the quantum phase operator in connection with the linear quantum canonical transformations (LCT). By this argument we imply that the algebraic formulation of the quantum phase problem is connected, through the Wigner-Weyl-Moyal (WWM) correspondence, with the existence of a canonical formalism of the quantum action-angle (AA) operators in the quantum phase space (QPS). This correspondence, although it will be shown to be manifest for arbitrary but finite dimensions leading to the finite dimensional algebraic realizations of the AA Wigner function, yields the desired correspondence between the quantum and the classical AA formalisms *only* in the transition to the continuum limit.

The main purpose of this article is to extend the canonical-algebraic approach to the quantum phase problem in Ref. [15] by formulating this correspondence explicitly in terms of the generators of the LCT. The quantum AA operators will be found in terms of the generators

of the LCT and it will be shown that the angle operator unitary-canonical to the quantum action will be identified as the unitary quantum phase operator.

Here we review some relevant parts of Ref. [15] for the completeness of the present work. Some additional material is also included in the appendix. The duality relations between the discrete generalized WK phase space operator basis $\Delta(\vec{n})$ and the Schwinger operator basis $\hat{S}_{\vec{m}}$ can be expressed as¹⁵

$$\hat{\Delta}(\vec{n}) = \frac{1}{D^{3/2}} \sum_{\vec{m}} e^{-i\gamma_0 (\vec{m} \times \vec{n})} \hat{S}_{\vec{m}} , \quad \hat{S}_{\vec{m}} = \frac{1}{\sqrt{D}} \sum_{\vec{n}} e^{i\gamma_0 (\vec{m} \times \vec{n})} \hat{\Delta}(\vec{n}) , \quad (1)$$

where $\vec{m} = (m_1, m_2)$, $\vec{n} = (n_1, n_2)$ are vectors in $\mathbb{Z}_D \times \mathbb{Z}_D$; $\vec{m} \times \vec{n} \equiv (m_1 n_2 - m_2 n_1)$, $\gamma_0 = 2\pi/D$, with D describing the dimension of the cyclic representations. Here the Schwinger operator basis $\hat{S}_{\vec{m}}$ is defined in terms of a finite dimensional unitary cyclic operator pair $(\hat{\mathcal{U}}, \hat{\mathcal{V}})$ such that

$$\hat{\mathcal{U}}^{m_1} \hat{\mathcal{V}}^{m_2} = e^{i\gamma_0 m_1 m_2} \hat{\mathcal{V}}^{m_2} \hat{\mathcal{U}}^{m_1} , \quad \hat{S}_{\vec{m}} = e^{-i\gamma_0 m_1 m_2 / 2} \hat{\mathcal{U}}^{m_1} \hat{\mathcal{V}}^{m_2} . \quad (2)$$

The D dimensional cyclic eigenspace $\{|v\rangle_k\}_{0 \leq k \leq (D-1)}$ and $\{|u\rangle_k\}_{0 \leq k \leq (D-1)}$ of the operators $\hat{\mathcal{U}}, \hat{\mathcal{V}}$ satisfy

$$\begin{aligned} \hat{\mathcal{U}} |v\rangle_k &= e^{i\gamma_0 k} |v\rangle_k , & |v\rangle_{k+D} &= |v\rangle_k \\ \hat{\mathcal{V}} |u\rangle_k &= e^{-i\gamma_0 k} |u\rangle_k , & |u\rangle_{k+D} &= |u\rangle_k \\ \hat{\mathcal{U}} |u\rangle_k &= |u\rangle_{k+1} \\ \hat{\mathcal{V}} |v\rangle_k &= |v\rangle_{k+1} \end{aligned} \quad (3)$$

and define a unitary Fourier duality as

$$\{|v\rangle\} = \hat{\mathcal{F}} \{|u\rangle\} , \quad \text{where} \quad (\hat{\mathcal{F}})_{k,k'} = \frac{1}{\sqrt{D}} e^{-i\gamma_0 k k'} , \quad \hat{\mathcal{F}}^\dagger = \hat{\mathcal{F}}^{-1} \quad (4)$$

where, the dual picture implies a Fourier automorphism on $\hat{\mathcal{U}}$ and $\hat{\mathcal{V}}$ in a sequence of transformations as

$$\begin{pmatrix} \hat{\mathcal{U}} \\ \hat{\mathcal{V}} \end{pmatrix} \xrightarrow{\hat{\mathcal{F}}} \begin{pmatrix} \hat{\mathcal{V}} \\ \hat{\mathcal{U}}^{-1} \end{pmatrix} \xrightarrow{\hat{\mathcal{F}}} \begin{pmatrix} \hat{\mathcal{U}}^{-1} \\ \hat{\mathcal{V}}^{-1} \end{pmatrix} \xrightarrow{\hat{\mathcal{F}}} \begin{pmatrix} \hat{\mathcal{V}}^{-1} \\ \hat{\mathcal{U}} \end{pmatrix} \xrightarrow{\hat{\mathcal{F}}} \begin{pmatrix} \hat{\mathcal{U}} \\ \hat{\mathcal{V}} \end{pmatrix} . \quad (5)$$

It can be shown that this Fourier operator duality between $\hat{\mathcal{U}}$ and $\hat{\mathcal{V}}$ implies

$$\hat{\mathcal{F}} \hat{S}_{\vec{m}} \hat{\mathcal{F}}^{-1} = \hat{S}_{R_{\pi/2} \cdot \vec{m}} , \quad \hat{\mathcal{F}}^4 = 1 , \quad \text{and} \quad R_{\pi/2}^4 = 1 \quad (6)$$

where $R_{\pi/2} : \vec{m} = (-m_2, m_1)$ corresponds to a $\pi/2$ rotation of the vector \vec{m} in the discrete phase space.

Eq's (2) and (3) imply for the properties of the $\hat{S}_{\vec{m}}$ basis

$$\begin{aligned} \hat{S}_{\vec{m}}^\dagger &= \hat{S}_{-\vec{m}} \\ \text{Tr}\{\hat{S}_{\vec{m}}\} &= D \delta_{\vec{m}, \vec{0}} \\ \hat{S}_{\vec{m}} \hat{S}_{\vec{m}'} &= e^{i \gamma_0 \vec{m} \times \vec{m}' / 2} \hat{S}_{\vec{m} + \vec{m}'} \\ (\hat{S}_{\vec{m}} \hat{S}_{\vec{m}'}) \hat{S}_{\vec{m}''} &= \hat{S}_{\vec{m}} (\hat{S}_{\vec{m}'} \hat{S}_{\vec{m}''}) \quad (\text{associativity}) \\ \hat{S}_{\vec{0}} &= \mathbb{I} \quad (\text{unit element}) \\ \hat{S}_{\vec{m}} \hat{S}_{-\vec{m}} &= \mathbb{I} \quad (\text{inverse}) . \end{aligned} \quad (7)$$

The generalized discrete Wigner function $W(\vec{n})$ in the physical state $|\psi\rangle \in H_D$ is defined by^{15,17}

$$W(\vec{n}) = \langle \psi | \hat{\Delta}(\vec{n}) | \psi \rangle \quad (8)$$

where Eq. (8) complies with all fundamental conditions that a generalized quantum phase space distribution should satisfy. The normalization of Eq. (8) is based on an appropriate summation of the WK operator basis in Eq. (1) over the discrete phase space vector \vec{n} . It is possible to use different normalizations when both(or one of the) labels are(is) continuous on the two dimensional torus. In these particular cases the phase space representations are based on $\mathbb{R} \times \mathbb{R}$ or $\mathbb{Z} \times \mathbb{R}$ respectively. Different normalizations are necessary for different choices of the phase space variables in order to obtain the appropriate continuum limit for the Wigner function. For instance, the symmetric normalization is necessary when the discrete phase space labels approach to the continuous ones simultaneously (i.e. $\mathbb{Z}_D \times \mathbb{Z}_D \rightarrow \mathbb{R} \times \mathbb{R}$) such as in the case of canonical pair of coordinate and momentum x, p leading to the continuous phase space distribution $W(x, p)$. The limit to continuous action-angle Wigner function $W(J, \theta)$ is recovered¹⁵ when one of the phase space labels is real and the other remains to be an integer in the limit $D \rightarrow \infty$; hence $\mathbb{Z}_D \times \mathbb{Z}_D \rightarrow \mathbb{Z} \times \mathbb{R}$. In section.II

we will base our formulation on the symmetric normalization as given in Eq. (1); whereas, in section 4, the AA Wigner function will be examined using the appropriate asymmetric normalization with $\mathbb{Z} \times \mathbb{R}$ without any loss of generality.

In section II we start the formulation of the LCT. The section II.A is devoted to the discrete scenario where the elements of LCT are in $SL(2, \mathbb{Z}_D)$. Their action on Schwinger's discrete cyclic operator basis is defined. The conditions of existence of a *unitary-canonical* partner to the generator of LCT are found for an arbitrary Hilbert space dimension and, provided such conditions are met, the existence of the unitary-canonical partner for each irreducible representation is demonstrated in the strong operator sense.

In section II.B the continuous scenario is examined. The elements of the continuous LCT are examined within the context of the irreducible representations of $SL(2, \mathbb{R})$. The operators corresponding to the unitary canonical partners of continuous LCT within each one parameter subgroup of $SL(2, \mathbb{R})$ as well as the entire group are derived by the matrix elements of the diagonal representations of the corresponding group elements. The section III is devoted to the Hamiltonian system with its dynamical symmetry group corresponding to the group of LCT. The unitary canonical partner to the generators of LCT is identified as the unitary phase operator and their equations of motion are derived separately for Hamiltonians with continuous as well as discrete spectrum. The connections between the quantum AA formalism and the dynamical symmetry is established at the *operator level*. The section IV is devoted to the construction of the AA-Wigner function. The continuous scenario is treated in section IV.a and the AA-Wigner function of the generalized oscillator with a discrete cyclic spectrum is presented in IV.b. The limit to the quantum harmonic oscillator AA formalism is also established.

II. GENERATORS OF THE LINEAR CANONICAL TRANSFORMATIONS

A. On the discrete toroidal lattice $\mathbb{Z}_D \times \mathbb{Z}_D$

The unitary Fourier automorphism in Eq. (5) implies the simplest discrete canonical transformation $\vec{m} \rightarrow R_{\pi/2} : \vec{m}$ on the phase space labels as given by Eq. (6). It was shown in Ref. [15] that Eq's (5) are a special case of a more general automorphic sequence produced by a unitary canonical transformation generator $\hat{\mathcal{G}}$ with $\hat{\mathcal{G}}^\dagger = \hat{\mathcal{G}}^{-1}$ where

$$\begin{aligned} \hat{\mathcal{G}}^\dagger \hat{U} \hat{\mathcal{G}} &= \hat{S}_{\vec{s}}, & \hat{\mathcal{G}}^\dagger \hat{V} \hat{\mathcal{G}} &= \hat{S}_{\vec{t}} \\ \hat{\mathcal{U}} &\xrightarrow{\hat{\mathcal{G}}} \hat{S}_{\vec{s}} \xrightarrow{\hat{\mathcal{G}}} \hat{S}_{s_1\vec{s}+s_2\vec{t}} \xrightarrow{\hat{\mathcal{G}}} \dots \\ \hat{\mathcal{V}} &\xrightarrow{\hat{\mathcal{G}}} \hat{S}_{\vec{t}} \xrightarrow{\hat{\mathcal{G}}} \hat{S}_{t_1\vec{s}+t_2\vec{t}} \xrightarrow{\hat{\mathcal{G}}} \dots \end{aligned} \quad (9)$$

Such a unitary generator satisfies

$$\hat{\mathcal{G}}^\dagger \hat{S}_{\vec{m}} \hat{\mathcal{G}} = \hat{S}_{R:\vec{m}}, \quad \text{where} \quad R : \vec{m} = \vec{m}' = (s_1 m_1 + t_1 m_2, s_2 m_1 + t_2 m_2) \quad (10)$$

with $\det R = \vec{s} \times \vec{t} = 1$ where $\vec{s} = (s_1, s_2)$ and $\vec{t} = (t_1, t_2)$ are two arbitrary labeling vectors in $\mathbb{Z}_D \times \mathbb{Z}_D$. Hence $R \in SL(2, \mathbb{Z}_D)$. Eq's (5) and (6) correspond to a special realization of Eq. (10) when $\vec{s} = (0, 1)$ and $\vec{t} = (-1, 0)$. The application of $\hat{\mathcal{G}}$ leaves Eq's (7) covariant.

Using Eq. (10) in Eq. (1) it can be shown that $\hat{\mathcal{G}}$ generates discrete canonical transformations in the WK basis as

$$\Delta(\vec{n}') = \hat{\mathcal{G}}^\dagger \Delta(\vec{n}) \hat{\mathcal{G}} = \Delta(R^{-1} : \vec{n}) \quad (11)$$

where $R^{-1} : \vec{n} = \vec{n}' = (t_2 n_1 - t_1 n_2, -s_2 n_1 + s_1 n_2)$.

The explicit form of $\hat{\mathcal{G}}$ and its irreducible representations have been studied in detail for the specific case of D being a prime of the type $D = 4k \pm 1$ where $k \in \mathbb{Z}$, in connection with the Schwinger operator basis by regarding $\hat{\mathcal{G}}$ as the generator of the time evolution (Hamiltonian) of the $SL(2, \mathbb{Z}_D)$ oscillator.²¹ Specifically, $\hat{\mathcal{G}}$ has cyclic generators which can be chosen as

$$g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} g_0 & 0 \\ 0 & g_0^{-1} \end{pmatrix}, \quad g_3 = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a^2 + b^2 = 1(\text{mod } D) \right\} \quad (12)$$

with periods D , $D - 1$ and $4k$ respectively. Here g_0 is a primitive element of \mathbb{Z}_D where $g_0^{D-1} = 1(\text{mod } D)$.

For such D , the explicit form of $\hat{\mathcal{G}}$ satisfying Eq's (9), or more compactly Eq. (10), is given by²¹

$$\hat{\mathcal{G}} = \hat{\mathcal{G}}(R) = \begin{cases} \frac{\sigma(1)\sigma(\delta)}{D} \sum_{\vec{m}} e^{i\gamma_0[t_1 m_1^2 + (t_2 - s_1)m_1 m_2 - s_2 m_2^2]/2\delta} \{m_1, m_2\}, & \text{if } \delta \neq 0, \\ \frac{\sigma(-2t_1)}{\sqrt{D}} \sum_{m_1} e^{i\gamma_0 m_1^2/2t_1} \{m_1(s_1 - 1)/t_1, m_1\}, & \text{if } \delta = 0, t_1 \neq 0, \\ \frac{\sigma(-2s_2)}{\sqrt{D}} \sum_{m_1} e^{-i\gamma_0 m_1^2/2s_2} \{m_1, 0\}, & \text{if } \delta = t_1 = 0, s_2 \neq 0 \end{cases} \quad (13)$$

where it is defined that $\delta \equiv 2 - s_1 - t_2$ and $\{m_1, m_2\} \equiv \hat{S}_{\vec{m}}$. Here $\sigma(m)$ is the Gauss sum

$$\sigma(m) \equiv \frac{1}{\sqrt{D}} \sum_{n=0}^{D-1} e^{i\gamma_0 m n^2} \quad (14)$$

It can be seen by direct inspection that $\hat{\mathcal{G}}(R^{-1}) = \hat{\mathcal{G}}^{-1}(R) = \hat{\mathcal{G}}^\dagger(R)$ namely $\hat{\mathcal{G}}$ is unitary.

Our main purpose in this section is to search for the condition of existence of a *unitary canonical* partner $\hat{\mathcal{O}}$ to $\hat{\mathcal{G}}$ such that

$$\hat{\mathcal{G}} \hat{\mathcal{O}} = \Omega \hat{\mathcal{O}} \hat{\mathcal{G}}, \quad |\Omega| = 1, \quad [\hat{\mathcal{G}}, \Omega] = [\hat{\mathcal{O}}, \Omega] = 0. \quad (15)$$

If Eq. (15) is satisfied for some pure phase factor Ω and a unitary $\hat{\mathcal{O}}$, then we consider Eq. (15) as a generalized canonical commutation relation for the pair $\hat{\mathcal{G}}, \hat{\mathcal{O}}$. The Eq. (15) then implies that $\hat{\mathcal{O}}$ rotates the eigenspectrum of $\hat{\mathcal{G}}$ in a cyclic order and visa versa. It is beyond the scope of the manuscript to examine the characterization of the irreducible representations of the most general group defined by $\hat{\mathcal{G}}, \hat{\mathcal{O}}, \Omega$ in Eq. (15) above. Here, we will confine our attention to those relatively simpler cases leading to the unique irreducible representations of the group in which the operator $\hat{\mathcal{O}}$ becomes the unitary-canonical partner of the generalized canonical transformation generator $\hat{\mathcal{G}}$. For this purpose, let us start with the simplest case such that for some non-zero integers a, b, c we have

$$\hat{\mathcal{G}}^a = \hat{\mathcal{O}}^b = \Omega^c = 1. \quad (16)$$

Then we call the group defined by the elements $\hat{\mathcal{G}}^{\ell_1} \hat{\mathcal{O}}^{\ell_2} \Omega^{\ell_3}$, where ℓ_1, ℓ_2, ℓ_3 are integers defined $(\text{mod } a)$, $(\text{mod } b)$, $(\text{mod } c)$ respectively, as a discrete Heisenberg-Weyl group $\Gamma(a, b, c)$.

An explicit calculation yields that, the group elements are uniquely defined only when c divides both a and b (i.e. $a = ca', b = cb'$ where $a', b' \in \mathbb{Z}$). Furthermore, from Eq. (16) and (15) we also have $\Omega^a = \Omega^b = \Omega^c = 1$ which implies that a, b, c must have a greatest common divisor d . These results imply that $a = dc''a', b = dc''b', c = dc''$ where $c'' \in \mathbb{Z}$. Without loss of generality we will assume that $c'' = 1$. The group defined by the Schwinger operator basis $S_{\vec{m}}$ in Eq's (2) and (3) with general elements as $\hat{\mathcal{U}}^{m_1} \hat{\mathcal{V}}^{m_2} \omega^{m_3}$ where m_1, m_2, m_3 are integers ($\text{mod } D$) is then a specific example of $\Gamma(a, b, c)$ with $a = b = c = D$. The number of irreducible representations of $\Gamma(a, b, c)$ with $a = da', b = db', c = d$ depends on the numbers a', b' . For $a' = b' = 1$ there is only one irreducible representation which is d -dimensional and is given by the Weyl matrices²²,

$$\hat{\mathcal{G}} \rightarrow \text{diag}(1, \Omega, \Omega^2, \dots, \Omega^{d-1}) \quad , \quad \hat{\mathcal{O}} \rightarrow \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad (17)$$

where a unitary $\hat{\mathcal{O}}$ satisfying Eq. (15) exists as given in Eq. (17). For $a', b' \neq 1$, the number of such irreducible representations is given by the product $a'b'$ and they are all d -dimensional. Up to a unitary equivalence, each irreducible representation is isomorphic to that in Eq. (17). The connection of direct product representation of $\Gamma(d, d, d)$ and its connection with the Chinese remainder theorem for the unique prime factorization of d was studied recently in Ref. [23]. Each prime factor represents an independent physical degree of freedom allowing the extension of the phase space formalism presented here to more than one degrees of freedom.^{16,17} A similar decomposition can also be done for the more general group $\Gamma(a, b, d)$ with a, b, d as defined above. The correspondence between the discrete QPS action-angle formalism with one degree of freedom and the classical one can be extended to more than one degrees of freedom at an algebraic level. Within the purpose of this article we will establish this correspondence only for the case with one degree of freedom and examine the larger degrees of freedom in a separate work.

There is already an extensive literature on the representations of the discrete canonical transformations induced by $\hat{\mathcal{G}} = SL(2, \mathbb{Z}_D)$. One particularly important limit in the discrete scenario is when $\hat{\mathcal{G}}$ is represented only by the rotational generator g_3 in Eq. (12) corresponding to the discrete fractional Fourier operator $\hat{\mathcal{F}}^{1/k} = \hat{\mathcal{G}}$ such that $\hat{\mathcal{G}}^{4k} = \mathbb{I}$. This limit has been examined in detail both from the formal quantum mechanical²⁴ and more applied, non-algebraic perspectives^{25,26}. For an arbitrary Hilbert space dimension the multiplicities of the four distinct eigenvalues of the fractional Fourier operator $\hat{\mathcal{F}}^{1/k}$ are not identical²⁵ and neither $\hat{\mathcal{F}}$ nor $\hat{\mathcal{F}}^{1/4k}$ has exact unitary-canonical partner in the sense of $\hat{\mathcal{O}}$ satisfying Eq. (15).

B. Generators of the infinitesimal canonical transformations in $\mathbb{R} \times \mathbb{R}$

The modular group $SL(2, \mathbb{Z}_D)$ does not have a proper continuous limit into $SL(2, \mathbb{R})$; hence, we cannot take the formal limit $D \rightarrow \infty$ in Eq. (10) to examine the continuous scenario. We will base the continuous representation of linear canonical transformations in $SL(2, \mathbb{R})$ in the formal sense on

$$\hat{\mathcal{G}}_{\infty}^{\dagger} \hat{S}_{\vec{\alpha}} \hat{\mathcal{G}}_{\infty} = \hat{S}_{\vec{\alpha}'} \quad (18)$$

where $\vec{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R}$ is a continuous phase space vector, $\hat{S}_{\vec{\alpha}}$ are elements of the continuous Schwinger operator basis¹⁶ and $\vec{\alpha}' = R : \vec{\alpha}$ with $R \in SL(2, \mathbb{R})$ indicate the transformation matrix with real elements. From here on we will be confined to the continuous scenario in which we can drop the subscript ∞ from the canonical transformation generators $\hat{\mathcal{G}}_{\infty}$. The three one-parameter subgroups g_j , ($j = 1, 2, 3$) of $sl(2, \mathbb{R})$, as conventionally represented by the three 2×2 matrices, correspond to

$$\Omega_1(\psi) = \begin{pmatrix} \cosh \psi/2 & \sinh \psi/2 \\ \sinh \psi/2 & \cosh \psi/2 \end{pmatrix}, \quad \Omega_2(\theta) = \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix}, \quad \Omega_3(\varphi) = \begin{pmatrix} e^{\varphi/2} & 0 \\ 0 & e^{-\varphi/2} \end{pmatrix} \quad (19)$$

where $\Omega_j \in g_j$, $-\infty < \psi < \infty$, $-\pi < \theta < \pi$, and $-\pi \leq \varphi \leq \pi$. A generic group element $g \in SL(2, \mathbb{R})$ can be parameterized as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{where} \quad \det g = 1 \quad (20)$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ being functions of ψ, θ, φ .

The three Hermitian operators \hat{K}_j , ($j = 1, 2, 3$) corresponding to the infinitesimal generators of the transformation in each subgroup respect the commutation relations

$$[\hat{K}_1, \hat{K}_2] = i \hat{K}_3, \quad [\hat{K}_2, \hat{K}_3] = i \hat{K}_1, \quad [\hat{K}_1, \hat{K}_3] = i \hat{K}_2. \quad (21)$$

We are particularly interested in the continuous irreducible representation of \hat{K}_j , ($j = 1, 2, 3$) in the canonical phase space parameterized by $\vec{\alpha}$ in Eq. (18). This particular representation of the generators is given by

$$\begin{aligned} \hat{K}_1 &= -i (\alpha_1 \partial_{\alpha_2} + \alpha_2 \partial_{\alpha_1})/2 \\ \hat{K}_2 &= -i (\alpha_1 \partial_{\alpha_2} - \alpha_2 \partial_{\alpha_1})/2 \\ \hat{K}_3 &= -i (\alpha_1 \partial_{\alpha_1} - \alpha_2 \partial_{\alpha_2})/2 \end{aligned} \quad (22)$$

where each irreducible representation acts on the Hilbert space of homogeneous polynomials of degree 2ℓ , and, a definite parity ϵ where $2\ell \in \mathbb{Z}$ with $\epsilon = \pm$ describing the odd(-) and even(+) parity. Hence we characterize those irreducible representations using the standard notation by $T_\chi(g_j)$ where $\chi = (\ell, \epsilon)$ and the sector of the Hilbert space they belong to by H_χ . Also, in the family $T_\chi(g_j)$ we are particularly interested in the diagonal representations of each generator in Eq. (22).

1. Diagonal representations of the j 'th subgroup g_j

We will describe the eigenvectors $|e_j^\chi(\gamma_j)\rangle$ in the diagonal representations $T_\chi(g_j)$ characterized by a particular j where $j = (1, 2, 3)$ with their corresponding projections on the canonical phase space $\langle \vec{\alpha} | e_j^\chi(\gamma_j) \rangle \equiv e_j^\chi(\vec{\alpha}, \gamma_j)$. Considering the simplest case of $\ell = 0$ first, $\langle \vec{\alpha} | e_j^\chi(\gamma_j) \rangle$ are given by

$$\begin{aligned} T_\chi(\Omega_1) : \quad e_1^\chi(\vec{\alpha}, \gamma_1) &= C_1 \left(\frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} \right)^{i\gamma_1}, \\ T_\chi(\Omega_2) : \quad e_2^\chi(\vec{\alpha}, \gamma_2) &= C_2 \left(\frac{\alpha_1 + i\alpha_2}{\alpha_1 - i\alpha_2} \right)^{\gamma_2}, \\ T_\chi(\Omega_3) : \quad e_3^\chi(\vec{\alpha}, \gamma_3) &= C_3 \left(\frac{\alpha_1}{\alpha_2} \right)^{i\gamma_3} \end{aligned} \quad (23)$$

where $\gamma_j \in \mathbb{R}$ and C_j 's are constants based on an appropriate normalization by the inner product $\langle e_j^x(\gamma_j) | e_j^x(\gamma'_j) \rangle = \delta(\gamma_j - \gamma'_j)$. Eq's (23) imply that

$$\hat{K}_j |e_j^x(\gamma_j)\rangle = \gamma_j |e_j^x(\gamma_j)\rangle . \quad (24)$$

Within each subgroup g_j , ($j = 1, 2, 3$) we now define the unitary subgroup elements $\hat{\mathcal{G}}_j \in g_j$ such that

$$\hat{\mathcal{G}}_j^{\Gamma_j} = e^{-i\Gamma_j \hat{K}_j} , \quad (25)$$

where $\Gamma_j \in \mathbb{R}$, ($j = 1, 2, 3$). The representations of Eq's (25) in terms of 2×2 matrices in the phase space $\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ are given by the Ω_j 's in Eq. (19), namely, the action of the each group element $\hat{\mathcal{G}}_j$ in Eq. (18) is given by

$$(\hat{\mathcal{G}}_j^{\Gamma_j})^\dagger \hat{S}_{\vec{\alpha}} \hat{\mathcal{G}}_j^{\Gamma_j} \equiv \hat{S}_{\alpha'} = \hat{S}_{\Omega_j: \vec{\alpha}} . \quad (26)$$

Within each subgroup g_j there exists, in the Schwinger sense¹⁵, a *special canonical* partner denoted by $\hat{\mathcal{O}}_j$ of $\hat{\mathcal{G}}_j$ such that

$$\hat{\mathcal{G}}_j^{\Gamma_j} \hat{\mathcal{O}}_j^{\zeta_j} = e^{-i\Gamma_j \zeta_j} \hat{\mathcal{O}}_j^{\zeta_j} \hat{\mathcal{G}}_j^{\Gamma_j} . \quad (27)$$

Eq. (27) implies that

$$\hat{\mathcal{G}}_j^{\Gamma_j} |e_j^x(\gamma_j)\rangle = e^{-i\Gamma_j \gamma_j} |e_j^x(\gamma_j)\rangle , \quad \hat{\mathcal{O}}_j^{\zeta_j} |e_j^x(\gamma_j)\rangle \sim |e_j^x(\gamma_j + \zeta_j)\rangle \quad (28)$$

where \sim sign in the equation on the right indicates that the equality holds up to an indeterminable phase factor which we consider to be irrelevant. Then Eq. (28) provides a representation for the operator $\hat{\mathcal{O}}_j$ in the diagonal representation of $\hat{\mathcal{G}}_j$. To complete the picture, we consider for each subgroup g_j , the diagonal representation of the $\hat{\mathcal{O}}_j$ operator such that

$$\hat{\mathcal{O}}_j^{\zeta_j} |f_j^x(\eta_j)\rangle = e^{i\zeta_j \eta_j} |f_j^x(\eta_j)\rangle . \quad (29)$$

From Eq's (27) the action of $\hat{\mathcal{G}}_j^{\Gamma_j}$'s on this basis can be found as

$$\hat{\mathcal{G}}_j^{\Gamma_j} |f_j^X(\eta_j)\rangle \sim |f_j^X(\eta_j + \Gamma_j)\rangle . \quad (30)$$

The \sim indicates again that there is an overall indeterminable phase which we ignore without any loss of generality. Then Eq's (27-30) completely determine, in the weak sense, the properties of the operator pair $(\hat{\mathcal{G}}, \hat{\mathcal{O}})$ within the Hilbert space of each subgroup g_j as three different realizations of the Schwinger operator basis. The connection between the eigenbasis $|e_j^X(\gamma_j)\rangle$ and $|f_j^X(\eta_j)\rangle$ is given by the Fourier transformation

$$|f_j^X(\eta_j)\rangle = \int d\gamma_j e^{-i\gamma_j \eta_j} |e_j^X(\gamma_j)\rangle , \quad |e_j^X(\gamma_j)\rangle = \int d\eta_j e^{i\eta_j \gamma_j} |f_j^X(\eta_j)\rangle \quad (31)$$

where in short notation $|f_j^X(\eta_j)\rangle = \hat{\mathcal{F}} |e_j^X(\gamma_j)\rangle$ with $(\hat{\mathcal{F}})_{\eta_j, \gamma_j} = \langle f_j^X(\eta_j) | e_j^X(\gamma_j) \rangle$ describing the matrix elements parameterized by (η_j, γ_j) of the unitary Fourier operator $\hat{\mathcal{F}}$. Via Eq. (31) a Fourier automorphism is implied between the two eigenspaces for each j as

$$|e_j^X(\gamma_j)\rangle \xrightarrow{\hat{\mathcal{F}}} |f_j^X(\eta_j)\rangle \xrightarrow{\hat{\mathcal{F}}} |e_j^X(-\gamma_j)\rangle \xrightarrow{\hat{\mathcal{F}}} |f_j^X(-\eta_j)\rangle \xrightarrow{\hat{\mathcal{F}}} |e_j^X(\gamma_j)\rangle . \quad (32)$$

2. Diagonal representations of the entire group g

We now shift our attention from the parameterization of the diagonal representations to those of the entire group g of which the three-parameter group element will be denoted by $\hat{\mathcal{G}}$. For convenience of the calculations we adopt the unitary canonical form for $\hat{\mathcal{G}}$ as²⁷

$$\hat{\mathcal{G}}^\Lambda = e^{-i\vec{\Lambda} \cdot \vec{K}} , \quad \vec{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3) , \quad \vec{K} = (\hat{K}_1, \hat{K}_2, \hat{K}_3) \quad (33)$$

where $\vec{\Lambda}$ is defined on the $sl(2, \mathbb{R})$ invariant group manifold characterized by the invariant $\Lambda^2 = \Lambda_1^2 + \Lambda_3^2 - \Lambda_2^2$. Adopting the particular parameterization $\Lambda_1 = \Lambda \sin a \cosh b$, $\Lambda_2 = \Lambda \sin a \sinh b$, and $\Lambda_3 = \Lambda \cos a$, Eq. (33) can be obtained, let's say, from $\hat{\mathcal{G}}_3^\Lambda$ in Eq. (25) by the unitary transformation

$$\hat{\mathcal{G}}^\Lambda = (\hat{\mathcal{T}}_{23}^{(-a,b)})^\dagger \hat{\mathcal{G}}_3^\Lambda \hat{\mathcal{T}}_{23}^{(-a,b)} , \quad \text{where} \quad \hat{\mathcal{T}}_{23}^{(-a,b)} = \hat{\mathcal{G}}_2^{-a} \hat{\mathcal{G}}_3^b . \quad (34)$$

Since $\hat{\mathcal{G}}^\Gamma$ in Eq. (33) is an element of $sl(2, \mathbb{R})$, the irreducible representations also act on the homogeneous polynomials of degree 2ℓ and parity $\epsilon = \pm$ in H_χ .

i) Continuous diagonal representations:

Similar to Eq's (24) and to the first set in Eq's (28) for the subgroups, we now seek the eigenvectors $|h^x(\gamma)\rangle \in H_\chi$ of $\hat{\mathcal{G}}^\Lambda$ such that

$$\hat{\mathcal{G}}^\Lambda |h^x(\gamma)\rangle = e^{iA} |h^x(\gamma)\rangle, \quad A \in \mathbb{R} \quad (35)$$

where A and $|h^x(\gamma)\rangle$ are to be found from the eigenproblem in Eq. (35). Eq. (34) suggests that

$$|h^x(\gamma)\rangle = (\hat{\mathcal{T}}_{23}^{(-a,b)})^\dagger |e_3^x(\gamma)\rangle \quad (36)$$

where $A = -\Lambda$ in Eq. (35). Hence $|h^x(\gamma)\rangle$ spans the eigenspace of the unitary operator in Eq. (33) with $\Lambda, \gamma \in \mathbb{R}$. The orthonormality of the eigenbasis $|h^x(\gamma)\rangle$ is guaranteed by the unitary transformation in Eq. (36) and the orthonormality of the eigenbasis $|e_3^x(\gamma)\rangle$. A phase space representation for $|h^x(\gamma)\rangle$ similar to Eq's (23) can be found by projecting it on the phase space vector $\vec{\alpha}$ as

$$\langle \vec{\alpha} | h^x(\gamma) \rangle \equiv h^x(\vec{\alpha}, \gamma) = \langle \vec{\alpha} | (\hat{\mathcal{T}}_{23}^{(-a,b)})^\dagger | e_3^x(\gamma) \rangle \quad \hat{\mathcal{T}}_{23}^{(-a,b)} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \Omega_3(b) \Omega_2(a) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \quad (37)$$

where $\Omega_2(a)$ and $\Omega_3(b)$ are implied by Eq. (19). The *unitary-canonical* partner of $\hat{\mathcal{G}}$ can be found similarly as it was done for the subgroups. Defining $\hat{\mathcal{O}}$ such that

$$\hat{\mathcal{G}}^\Lambda \hat{\mathcal{O}}^\zeta = e^{-i\Lambda\zeta} \hat{\mathcal{O}}^\zeta \hat{\mathcal{G}}^\Lambda \quad (38)$$

we find

$$\begin{aligned} \hat{\mathcal{G}}^\Lambda |h^x(\gamma)\rangle &= e^{-i\Lambda\gamma} |h^x(\gamma)\rangle, & \hat{\mathcal{O}}^\zeta |h^x(\gamma)\rangle &\sim |h^x(\gamma + \zeta)\rangle \\ \hat{\mathcal{O}}^\zeta |k^x(\eta)\rangle &= e^{i\zeta\eta} |k^x(\eta)\rangle, & \hat{\mathcal{G}}^\Lambda |k^x(\eta)\rangle &\sim |k^x(\eta + \Lambda)\rangle \end{aligned} \quad (39)$$

where we will again neglect the overall phases in the second column of the relations above. The basis vectors in Eq. (39) are connected by the Fourier transformation

$$|k^x(\eta)\rangle = \int d\gamma e^{-i\gamma\eta} |h^x(\gamma)\rangle, \quad |h^x(\gamma)\rangle = \int d\eta e^{i\gamma\eta} |k^x(\eta)\rangle \quad (40)$$

and a similar automorphism to Eq. (32) at the vector level between $|h^x(\gamma)\rangle$ and $|k^x(\eta)\rangle$ as well as to Eq. (5) at the operator level between $\hat{\mathcal{G}}$ and $\hat{\mathcal{O}}$ can be written.

ii) Discrete diagonal representations:

In examining the discrete representations of the entire group we start with the diagonal ones $|e_2^X(m)\rangle$ of \hat{K}_2 and associate with them the eigenfunctions

$$\langle \vec{\alpha} | e_2^X(m) \rangle \equiv e_2^X(\vec{\alpha}; m) = N_2^{(\ell)} (\alpha_1^2 + \alpha_2^2)^\ell \left(\frac{\alpha_1 + i\alpha_2}{\alpha_1 - i\alpha_2} \right)^m \quad (41)$$

where $N_2^{(\ell)}$ is a normalization based on an inner product $\langle e_2^X(\vec{\alpha}; m) | e_2^X(\vec{\alpha}; m') \rangle = \delta_{m,m'}$. For the state in Eq. (41) we have

$$\begin{aligned} \hat{K}_2 |e_2^X(m)\rangle &= m |e_2^X(m)\rangle \\ \hat{K}_+ |e_2^X(m)\rangle &= -(\ell - m) |e_2^X(m+1)\rangle \\ \hat{K}_- |e_2^X(m)\rangle &= (\ell + m) |e_2^X(m-1)\rangle \end{aligned} \quad (42)$$

where $\hat{K}_\pm = (\hat{K}_1 \pm i\hat{K}_3)$ and the Casimir element $\hat{K}^2 = 1/2(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+) - K_2^2$ has eigenvalue $\ell(\ell+1)$. In terms of the $\ell \neq 0$ representation in Eq. (41) and (42), the continuous representations that we used in Eq. (23) correspond to the $\ell = 0$ case. A general eigenvector for a group element in Eq. (33) can be found similarly as in the continuous case by redefining $\vec{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3)$ as, for instance, $\Lambda_1 = \Lambda \sinh a' \sin b'$, $\Lambda_2 = \Lambda \cosh a'$, $\Lambda_3 = \Lambda \sinh a' \cos b'$ and the transformation

$$\hat{\mathcal{G}}^\Lambda = (\hat{\mathcal{T}}_{12}^{(a',b')})^\dagger \hat{\mathcal{G}}_2^\Lambda \hat{\mathcal{T}}_{12}^{(a',b')} , \quad \text{where} \quad \hat{\mathcal{T}}_{12}^{(a',b')} = \hat{\mathcal{G}}_1^a \hat{\mathcal{G}}_2^b . \quad (43)$$

The eigenvectors $|h^X(m)\rangle$ of a general group element $\hat{\mathcal{G}}^\Lambda$ are then associated with the functions in H_X

$$\langle \vec{\alpha} | h^X(m) \rangle \equiv h^X(\vec{\alpha}; m) = \langle \vec{\alpha} | (\hat{\mathcal{T}}_{12}^{(a',b')})^\dagger | e_2^X(m) \rangle , \quad \hat{\mathcal{T}}_{12}^{(a',b')} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \Omega_2(-b') \Omega_1(a') \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \quad (44)$$

where $\Omega_2(-b)$ and $\Omega_1(a)$ are implied by Eq's (19). To establish the representation of the unitary canonical pair $(\hat{\mathcal{G}}, \hat{\mathcal{O}})$ in the discrete case we consider

$$\hat{\mathcal{G}}^\Lambda \hat{\mathcal{O}}^r = e^{-i\Lambda r} \hat{\mathcal{O}}^r \hat{\mathcal{G}}^\Lambda \quad (45)$$

corresponding to a change $\zeta \rightarrow r \in \mathbb{Z}$ in Eq. (38). For the system of eigenvectors we find

$$\begin{aligned}
\hat{\mathcal{G}}^\Lambda |h^\chi(m)\rangle &= e^{-i\Lambda m} |h^\chi(m)\rangle , & \hat{\mathcal{O}}^r |h^\chi(m)\rangle &\sim |h^\chi(m+r)\rangle \\
\hat{\mathcal{O}}^r |k^\chi(\eta)\rangle &= e^{i\eta r} |k^\chi(j, \eta)\rangle , & \hat{\mathcal{G}}^\Lambda |k^\chi(\eta)\rangle &\sim |k^\chi(\eta+\Lambda)\rangle
\end{aligned} \tag{46}$$

where $\Lambda, \eta \in \mathbb{R}$. The eigenspaces are connected by the (discrete and continuous) Fourier transformation as

$$k^\chi(\vec{\alpha}; \eta) = \sum_m e^{-i\eta m} h^\chi(\vec{\alpha}; m) , \quad h^\chi(\vec{\alpha}; m) = \int d\eta e^{im\eta} k^\chi(\vec{\alpha}; \eta) . \tag{47}$$

The Eq's (46) and (47) conclude our brief treatment of the unitary continuous phase space representations of the canonical $\hat{\mathcal{G}}, \hat{\mathcal{O}}$ pair in the weak matrix element sense.

III. IMPLICATIONS FOR THE HAMILTONIAN SYSTEMS WITH DYNAMICAL SYMMETRY GROUP AND GENERALIZED ACTION-ANGLE OPERATORS

Recently an algebraic approach was studied by Wang and Chu²⁸ in the solution of the one dimensional inverse problem for the Hamiltonian systems with dynamical group symmetry. In the simplest case of one dimension, the inverse problem reduces to that of finding an operator regarding the dynamical symmetry group of the system such that it will be invariant under either a certain subgroup or the entire group of canonical transformations. The group of canonical transformations then naturally reflects the properties of the dynamical symmetry of the system. For one dimensional *autonomous* systems, the invariant operator under canonical transformations corresponds to the generalized *action* operator, by which the Hamiltonian of the entire system can be fully described. In the following, we will examine the quantum canonical transformation group as the dynamical symmetry group for an Hamiltonian system and derive the equations of motion describing the time evolution of the generalized quantum action-angle operators.

Let us assume that the Hamiltonian $\hat{\mathcal{H}}$ describing the dynamics of a system with one degree of freedom is represented by

$$\hat{\mathcal{H}} \equiv \mathcal{H}(\hat{K}_1, \hat{K}_2, \hat{K}_3) \tag{48}$$

where \hat{K}_j 's are the Hermitian generators of the $sl(2, \mathbb{R})$ canonical transformation algebra in Eq. (33). The simplest but a sufficiently general example of an Hamiltonian with a dynamical group symmetry can then be obtained if the Hamiltonian is a real function of the linear superposition of \hat{K}_j 's as,

$$\hat{\mathcal{H}} = \mathcal{H}(P \hat{K}_1 + Q \hat{K}_2 + R \hat{K}_3) , \quad P, Q, R \in \mathbb{R} . \quad (49)$$

Here, we do not make any assumptions aside from the one that \mathcal{H} is a real valued and a well-behaved function. Our focus will be on those unitary elements of the dynamical symmetry group that can be described by

$$\hat{\mathcal{G}}^\Gamma = e^{-i\Gamma(P \hat{K}_1 + Q \hat{K}_2 + R \hat{K}_3)} \quad (50)$$

where the Hamiltonian in Eq. (49) becomes

$$\hat{\mathcal{H}} = \mathcal{H}(i \frac{\partial}{\partial \Gamma}) \hat{\mathcal{G}}^\Gamma \Big|_{\Gamma=0} . \quad (51)$$

The time dependent eigenvectors $|\psi_\gamma^\chi(t)\rangle$ of $\hat{\mathcal{H}}$ and the eigenenergies in the continuous representation are given by

$$\langle \vec{\alpha} | \psi_\gamma^\chi(t) \rangle = e^{iE_\gamma t} h^\chi(\vec{\alpha}, \gamma) , \quad E_\gamma = \mathcal{H}(\gamma) \quad (52)$$

where $h^\chi(\vec{\alpha}, \gamma)$ is given by Eq. (37). Here, the unitary canonical partner $\hat{\mathcal{O}}$ to $\hat{\mathcal{G}}$ plays the role of the unitary angle operator and is defined by Eq. (38). The time evolution of $\hat{\mathcal{O}}^\zeta$ is given by

$$i \frac{d}{dt} \langle \psi_{\gamma'} | \hat{\mathcal{O}}^\zeta | \psi_\gamma \rangle = \langle \psi_{\gamma'} | [\hat{\mathcal{H}}, \hat{\mathcal{O}}^\zeta] | \psi_{\gamma'} \rangle = -\langle \psi_{\gamma'} | \hat{\mathcal{O}}^\zeta | \psi_\gamma \rangle \{ \mathcal{H}(\gamma + \zeta) - \mathcal{H}(\gamma) \} . \quad (53)$$

If we symbolically associate an *Hermitian* phase operator at $\zeta = 0$ with

$$\hat{\phi} = -i \frac{d}{d\zeta} \hat{\mathcal{O}}^\zeta \Big|_{\zeta=0} \quad (54)$$

Eq. (53) becomes, in terms of $\hat{\phi}$

$$\lim_{\zeta \rightarrow 0} \langle \psi_{\gamma-\zeta} | \frac{d}{dt} \hat{\phi} | \psi_\gamma \rangle = -\frac{dE_\gamma}{d\gamma} \lim_{\zeta \rightarrow 0} \langle \psi_{\gamma-\zeta} | \psi_\gamma \rangle \quad (55)$$

which is the quantum analog of the classical equation of motion for the canonical angle variable. Hence the unitary operator $\hat{\mathcal{O}}$ should be considered as the *quantum angle operator* which is the unitary canonical partner of the action operator $\hat{\mathcal{G}}$.

The properties of the dynamical group symmetry can also be examined by using the discrete representation. In this case the Eq. (51) is still valid whereas (52) becomes

$$\langle \vec{\alpha} | \psi_m^x(t) \rangle = e^{iE_m t} h^x(\vec{\alpha}; m), \quad E_m = \mathcal{H}(m) \quad (56)$$

where $\langle \vec{\alpha} | \psi_m^x(t) \rangle$ are the time dependent eigenvectors, $h^x(\vec{\alpha}; m)$ is given by Eq. (44) and $E_m = \mathcal{H}(m)$ is the discrete eigenenergy spectrum depending on the discrete eigenstate index m . The appropriate angle operator $\hat{\mathcal{O}}^r$ in the discrete case has been studied in Eq. (45) and (46). The nature of the operator $\hat{\mathcal{O}}^r$ becomes more transparent if we examine the derivative of Eq. (45) with respect to Λ at $\Lambda = 0$. This can be readily evaluated as

$$\frac{d}{d\Lambda} \rightarrow \text{Eq. (45)} \Big|_{\Lambda=0} \implies [\vec{n} \cdot \vec{\hat{K}}, \hat{\mathcal{O}}^r] = -r \hat{\mathcal{O}}^r \quad (57)$$

which is the generalized Susskind-Glogower-Carruthers-Nieto commutation relation^{5,7,15} for the generalized *radial number operator* $\vec{n} \cdot \vec{\hat{K}}$ and the generalized unitary phase operator $\hat{\mathcal{O}}$. The time evolution of $\hat{\mathcal{O}}^r$ is then

$$i \frac{d}{dt} \langle \psi_{m'}^x | \hat{\mathcal{O}}^r | \psi_m^x \rangle = \langle \psi_{m'}^x | [\hat{\mathcal{H}}, \hat{\mathcal{O}}^r] | \psi_m^x \rangle = -\langle \psi_{m'}^x | \hat{\mathcal{O}}^r | \psi_m^x \rangle \{ \mathcal{H}(m+r) - \mathcal{H}(m) \} \quad (58)$$

which is the equation of motion for the unitary canonical angle operator of the generalized oscillator with a discrete spectrum $E_m = \mathcal{H}(m)$. Here, two results are in order. The first is that we have found a correspondence between the classical and quantum AA formalisms for Hamiltonian systems with a dynamical group symmetry of the type given by Eq. (49) and (50). The second result is the equivalence of the quantum action operator to the generators of canonical transformations as well as that of the unitary-canonical angle operator to the unitary-canonical phase operator.

Although the continuous and discrete representations of the AA operators are similar, depending on the continuous/discrete nature of the eigenenergy spectrum, one or the other is

more convenient in the formulation of a physical problem. This will be more transparent in the next section when we discuss the action-angle formalism of a generalized oscillator.

A. Implications for the generalized oscillator Hamiltonian and the quantum phase operator

In this and the following sections we will refer to the appendix which includes some relevant parts of Ref. [15]. There, it was shown that the quantum harmonic oscillator (QHO) algebra is recovered in the infinite dimensional limit (i.e. $D \rightarrow \infty$ hence $q \rightarrow 1$) of the admissible q-oscillator algebraic realization in Eq's (A.4). The importance of the naturally emerging admissible q-oscillator realizations is that they admit an algebraic formulation of the quantum phase problem and also provide a natural basis to examine the harmonic oscillator phase in the infinite dimensional limit of the algebra in Eq's (A.4). Respecting the historical development, we will nevertheless start with a brief outline of the phase problem using the dynamical continuous symmetry group of the QHO. The generators \hat{K}_i , ($i = 1, 2, 3$) of the dynamical $sl(2, \mathbb{R})$ symmetry of the QHO in the \hat{x}, \hat{p} representation are given by

$$\hat{K}_1 = \frac{1}{4} (\hat{x}^2 - \hat{p}^2), \quad \hat{K}_2 = \frac{1}{4} (\hat{x}^2 + \hat{p}^2), \quad \hat{K}_3 = \frac{1}{4} (\hat{x}\hat{p} + \hat{p}\hat{x}), \quad [\hat{x}, \hat{p}] = i \quad (59)$$

where the generators respect Eq's (21). Our first attempt will be to find the unitary canonical partner to \hat{K}_3 in Eq's (59). With $\hat{p} \rightarrow -i\partial/\partial x$, the eigenproblem for \hat{K}_3 yields

$$\hat{K}_3 |\psi_3(\gamma_3)\rangle = \gamma_3 |\psi_3(\gamma_3)\rangle, \quad \gamma_3 \in \mathbb{R}, \quad \langle x | \psi_3(\gamma_3) \rangle = N_3 x^{(2i\gamma_3 - 1/2)} \quad (60)$$

where N_3 is a normalization based on an inner product. Hence, for $\hat{\mathcal{G}}_3^{\Gamma_3} = e^{-i\Gamma_3 \hat{K}_3}$,

$$\hat{\mathcal{G}}_3^{\Gamma_3} |\psi_3(\gamma_3)\rangle = e^{-i\Gamma_3 \gamma_3} |\psi_3(\gamma_3)\rangle. \quad (61)$$

The unitary canonical partner $\hat{\mathcal{O}}_3$ to $\hat{\mathcal{G}}_3$, such that Eq. (27) is satisfied for $i = 3$, can be found following the steps leading to Eq's (28-30). The $\hat{\mathcal{O}}_3^\zeta$ operator for an arbitrary and real ζ is given by

$$\begin{aligned}
\hat{\mathcal{O}}_3^\zeta &= \int_{-\infty}^{\infty} d\gamma_3 |\psi(\gamma_3 + \zeta)\rangle \langle \psi(\gamma_3)|, & \text{or equivalently by} \\
\hat{\mathcal{O}}_3^\zeta &= \int_{-\infty}^{\infty} d\eta_3 e^{i\zeta \eta_3} |\tilde{\psi}(\eta_3)\rangle \langle \tilde{\psi}(\eta_3)|, & \text{where} \\
|\tilde{\psi}(\eta_3)\rangle &= \int_{-\infty}^{\infty} d\gamma_3 e^{-i\eta_3 \gamma_3} |\psi(\gamma_3)\rangle
\end{aligned} \tag{62}$$

namely, the existence of $\hat{\mathcal{O}}_3$ is manifested by the presence of a complete spectrum of \hat{K}_3 on the real axis and, in return, $\hat{\mathcal{O}}_3$ and $\hat{\mathcal{G}}_3$ are connected by a Fourier automorphism. A similar procedure can also be applied to \hat{K}_1 in Eq. (59) since the eigenspectrum of this operator also spans the symmetric positive and negative values on the entire real axis. However, there is a problem with the \hat{K}_2 operator. Because of the fact that \hat{K}_2 in Eq's (59) is a non-negative operator its eigenspectrum spans only the positive real axis. Hence, the Fourier automorphism is not applicable to \hat{K}_2 and in return, the unitary canonical partner to $\hat{\mathcal{G}}_2$ cannot be found. This problem has been attacked from a completely different perspective a long time ago in the elegant work of Boyer and Wolf¹¹ where they made use of the unitary isomorphism between the *radial* representation of the dynamical $SL(2, \mathbb{R})$ symmetry group of the multi-dimensional QHO with an added centrifugal term of arbitrary strength and the representation of the same group on the unit circle. Through this unitary mapping the space of square integrable functions on the unit circle is an inner product space endowed with a translationally invariant non-local measure. However, the drawback of this elegant method is that, the unitary irreducible representations are not single-valued under full rotations on the unit circle and this applies particularly to the standard one-dimensional quantum harmonic oscillator.

The phase problem in the QHO being the central theme of this work, we suggest here and in the following sections an alternative and, perhaps, a formally simpler way of looking into the phase problem in the QHO. We will now start with the admissible q-oscillator realization in Eq. (A.4) with the operators \hat{A} , \hat{A}^\dagger and \hat{Q} and approach the QHO algebra by extending D to infinity. We formally express the generalized oscillator Hamiltonian as [in analogy with Eq. (51)]

$$\hat{\mathcal{H}} = \mathcal{H}(\hat{N}) = \mathcal{H}(q \partial / \partial q) \hat{Q} \Big|_{q \rightarrow 1} \tag{63}$$

where the limit $q \rightarrow 1$ is achieved simultaneously with $D \rightarrow \infty$. The deformed algebra defined by the elements \hat{A} , \hat{A}^\dagger and \hat{Q} is an admissible version of the well-known (deformed) q -oscillator algebra naturally admitting real and non-negative norms in the finite dimensional cyclic Hilbert space H_D . Since the deformation parameter $q = e^{-i\gamma_0 \vec{m} \times \vec{m}'}$ is a pure phase with the property that $q^D = 1$, the operators \hat{A} , \hat{A}^\dagger , \hat{Q} act on in the finite- D -dimensional cyclic Fock space spanned by the cyclic orthonormal vectors $\{|n\rangle\} = \{|n\rangle_{0 \leq n \leq (D-1)}; |n\rangle = |n+D\rangle\}$ with $\langle n'|n\rangle = \delta_{n',n}$ as,

$$\begin{aligned}\hat{A}|n\rangle &= \sqrt{f(n)}|n-1\rangle \\ \hat{A}^\dagger|n\rangle &= \sqrt{f(n+1)}|n+1\rangle \\ \hat{Q}|n\rangle &= q^n|n\rangle, \quad q = e^{-i\gamma_0 \vec{m} \times \vec{m}'}\end{aligned}\tag{64}$$

with $0 \leq f(n)$ and $f(n) = f(n+D)$ where

$$f(n) = \frac{q^{n+(D-1)/2} - q^{-n-(D-1)/2}}{q - q^{-1}} + C, \quad C = \frac{2}{|q - q^{-1}|} \neq 0. \tag{65}$$

The algebra in Eq's (A.4) and the relations (64) admit a unitary canonical partner to \hat{Q} , i.e. the unitary quantum phase operator $\hat{\mathcal{E}}_\phi$

$$\hat{\mathcal{E}}_\phi = \sum_{n=0}^{D-1} |n-1\rangle\langle n|, \quad \hat{\mathcal{E}}_\phi^D \equiv \mathbb{I} \tag{66}$$

such that

$$\hat{Q}^\Gamma \hat{\mathcal{E}}_\phi^\lambda = q^{\Gamma\lambda} \hat{\mathcal{E}}_\phi^\lambda \hat{Q}^\Gamma, \quad \Gamma, \lambda \in \mathbb{R}. \tag{67}$$

The eigenvectors of $\hat{\mathcal{E}}_\phi$ are $\{|\phi\rangle\} = \{|\phi\rangle_{0 \leq r \leq (D-1)}; |\phi\rangle_{r+D} \equiv |\phi\rangle_r\} \in H_D$ with ${}_r\langle\phi|\phi\rangle_r = \delta_{r',r}$ where

$$\hat{\mathcal{E}}_\phi |\phi\rangle_r = e^{i\gamma_0 r} |\phi\rangle_r, \quad |\phi\rangle_r = \frac{1}{\sqrt{D}} \sum_{n=0}^{D-1} e^{i\gamma_0 nr} |n\rangle, \quad \gamma_0 = \frac{2\pi}{D}. \tag{68}$$

The dynamical time evolution of $\hat{\mathcal{E}}_\phi$ for the generalized Hamiltonian in Eq. (63) is given by Eq. (58) as

$$\begin{aligned}i \frac{d}{dt} \langle n' | \hat{\mathcal{E}}_\phi^r | n \rangle &= -\langle n' | \hat{\mathcal{E}}_\phi^r | n \rangle \left\{ \mathcal{H}(q \partial / \partial q) (q^{n+r} - q^n) \Big|_{q \rightarrow 1} \right\} \\ &= -\langle n' | \hat{\mathcal{E}}_\phi^r | n \rangle \left\{ \mathcal{H}(n+r) - \mathcal{H}(n) \right\}\end{aligned}\tag{69}$$

and a close inspection of Eq. (69) with (58) indicates that the pair $(\hat{Q}, \hat{\mathcal{E}}_\phi)$ is the corresponding *unitary action-angle pair* for the Hamiltonian in Eq. (63). As mentioned before, the $q \rightarrow 1$ and the $D \rightarrow \infty$ limits are to be taken simultaneously on both sides of Eq. (69). In this limit the phase operator $\hat{\mathcal{E}}_\phi$ is the *unitary version* of the Hermitian Pegg-Barnett phase operator²⁹. Making use of the fact that $\hat{Q} = q^{\hat{N}}$, Eq. (67) yields the Susskind-Glogower-Carruthers-Nieto commutation relations^{5,7,15} for the operator pair $(\hat{N}, \hat{\mathcal{E}}_\phi)$. Note that, the QHO described by $\mathcal{H} = \omega n$ yields the equation of motion for the phase operator in Eq. (69) which is formally identical to the equation of motion of the canonical angle variable for the classical harmonic oscillator. When we examine the quantum action-angle formalism of the generalized quantum oscillator in section IV.B, the harmonic oscillator will be realized in the specific limit when the dimension of the discrete phase space representations of the generalized oscillator is extended to infinity.

IV. AN EQUIVALENT REALIZATION OF THE WIGNER FUNCTION BY $(\hat{\mathcal{G}}, \hat{\mathcal{O}})$

A. A generalized approach to the AA Wigner function using continuous phase space representations

It was shown that the generalized canonical phase space representation of a quantum system can be given based on the duality between the discrete Wigner-Kirkwood and the unitary cyclic Schwinger operator bases^{15,16} in Eq. (1). An alternative to this approach is to formulate the same problem using the canonical transformation $\hat{\mathcal{G}}$ and its unitary canonical partner $\hat{\mathcal{O}}$.

The properties of $(\hat{\mathcal{G}}, \hat{\mathcal{O}})$ studied in sections II and III manifest a full analogy to those of $(\hat{\mathcal{U}}, \hat{\mathcal{V}})$ in Eq's (3-5). Among the four equivalent choices in (5), we define this analogy by the correspondence

$$\begin{pmatrix} \hat{\mathcal{U}} \\ \hat{\mathcal{V}} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \hat{\mathcal{O}} \\ \hat{\mathcal{G}} \end{pmatrix}. \quad (70)$$

It is now suggestive to define a Schwinger operator basis labeled by $\vec{\tau} = (\tau_1, \tau_2) \in \mathbb{R} \times \mathbb{R}$ and defined as

$$\hat{\Sigma}_{\vec{\tau}} = e^{-i\tau_1\tau_2/2} \hat{\mathcal{O}}^{\tau_1} \hat{\mathcal{G}}^{\tau_2} = e^{i\tau_1\tau_2/2} \hat{\mathcal{G}}^{\tau_2} \hat{\mathcal{O}}^{\tau_1} , \quad (71)$$

Before we study the algebraic properties of $\hat{\Sigma}_{\vec{\tau}}$, we look into some of the tracial properties of $\hat{\mathcal{G}}$ and $\hat{\mathcal{O}}$ operators. Since we consider the continuum limit, it is more appropriate to examine $\hat{\mathcal{G}}$ and $\hat{\mathcal{O}}$ in their continuous representation. Starting with Eq's (39) we choose the $|h^\chi(\gamma)\rangle$ basis for their representation as

$$\hat{\mathcal{O}}^{\tau_1} = \int_{-\infty}^{\infty} d\gamma |h^\chi(\gamma + \tau_1)\rangle \langle h^\chi(\gamma)| , \quad \hat{\mathcal{G}}^{\tau_2} = \int_{-\infty}^{\infty} d\gamma e^{-i\tau_2\gamma} |h^\chi(\gamma)\rangle \langle h^\chi(\gamma)| \quad (72)$$

from which we obtain

$$Tr\{\hat{\mathcal{O}}^{\tau_1} \hat{\mathcal{G}}^{\tau_2}\} = \int_{-\infty}^{\infty} d\gamma' \langle h^\chi(\gamma') | \{\hat{\mathcal{O}}^{\tau_1} \hat{\mathcal{G}}^{\tau_2}\} | h^\chi(\gamma') \rangle = 2\pi \delta(\tau_1) \delta(\tau_2) \equiv 2\pi \delta(\vec{\tau}) . \quad (73)$$

Using Eq. (73) the properties of $\hat{\Sigma}_{\vec{\tau}}$ can be found in manifest analogy with those of $\hat{S}_{\vec{m}}$ in Eq's (7) as

$$\begin{aligned} \hat{\Sigma}_{\vec{\tau}}^\dagger &= \hat{\Sigma}_{-\vec{\tau}} \\ Tr\{\hat{\Sigma}_{\vec{\tau}}\} &= 2\pi \delta(\vec{\tau}) \\ \hat{\Sigma}_{\vec{\tau}} \hat{\Sigma}_{\vec{\tau}'} &= e^{i\vec{\tau} \times \vec{\tau}' / 2} \hat{\Sigma}_{\vec{\tau} + \vec{\tau}'} \\ (\hat{\Sigma}_{\vec{\tau}} \hat{\Sigma}_{\vec{\tau}'}) \hat{\Sigma}_{\vec{\tau}''} &= \hat{\Sigma}_{\vec{\tau}} (\hat{\Sigma}_{\vec{\tau}'} \hat{\Sigma}_{\vec{\tau}''}) \quad (\text{associativity}) \\ \hat{\Sigma}_{\vec{0}} &= \mathbb{I} \quad (\text{unit element}) \\ \hat{\Sigma}_{\vec{\tau}} \hat{\Sigma}_{-\vec{\tau}} &= \mathbb{I} \quad (\text{inverse}) . \end{aligned} \quad (74)$$

Hence, the canonical transformation generator $\hat{\mathcal{G}}$ and its unitary canonical partner $\hat{\mathcal{O}}$ form a continuous realization of Schwinger's operator basis.

Using Eq (1) and the analogy manifested by Eq (70) we construct a dual form for the Wigner-Kirkwood operator basis using the realization of the Schwinger basis defined in Eq's (71) and (74) as

$$\hat{\Delta}_{CT}(\vec{V}) = \int \frac{d\vec{\tau}}{2\pi} e^{-i\vec{\tau} \times \vec{V}} \hat{\Sigma}_{\vec{\tau}} , \quad \hat{\Sigma}_{\vec{\tau}} = \int \frac{d\vec{V}}{2\pi} e^{i\vec{\tau} \times \vec{V}} \hat{\Delta}_{CT}(\vec{V}) \quad (75)$$

where the integrals are to be considered in $\mathbb{R} \times \mathbb{R}$. Using the properties of $\hat{\Sigma}_{\vec{\tau}}$ it can be shown that Eq's (75) provide an operator basis for the Wigner function as

$$\begin{aligned}\hat{\Delta}_{CT}(\vec{V}) &= \hat{\Delta}_{CT}^\dagger(\vec{V}) \\ \int \frac{d\vec{V}}{2\pi} \hat{\Delta}_{CT}(\vec{V}) &= \mathbb{I} \\ Tr\{\hat{\Delta}_{CT}(\vec{V})\} &= \mathbb{I} \\ Tr\{\hat{\Delta}_{CT}(\vec{V}) \hat{\Delta}_{CT}(\vec{V}')\} &= \delta(\vec{V} - \vec{V}') .\end{aligned}\tag{76}$$

The properties (76) are necessary and sufficient conditions in order to define a correspondence between an arbitrary operator \hat{F} and its Weyl-Wigner-Moyal (WWM) symbol $f(\vec{V})$

$$\hat{F} = \int d\vec{V} f(\vec{V}) \hat{\Delta}_{CT}(\vec{V}) , \quad f(\vec{V}) = Tr\{\hat{F} \hat{\Delta}_{CT}(\vec{V})\}\tag{77}$$

with the condition that $\|\hat{F}\| = \int d\vec{V} |f(\vec{V})|^2 < \infty$. A few simple examples can be given

a) for $\hat{F} = \hat{\mathcal{G}}^\Lambda$

Using the same relations as in (a), the WWM symbol of $\hat{\mathcal{G}}^\Lambda$ is

$$g_\Lambda(\vec{V}) = Tr\{\hat{\mathcal{G}}^\Lambda \hat{\Delta}_{CT}(\vec{V})\} = e^{-i\Lambda V_1}\tag{78}$$

and similarly

b) for $\hat{F} = \hat{\mathcal{O}}^\zeta$

$$o_\zeta(\vec{V}) = Tr\{\hat{\mathcal{O}}^\zeta \hat{\Delta}_{CT}(\vec{V})\} = e^{i\zeta V_2}\tag{79}$$

c) A particularly interesting case arises when the arbitrary operator \hat{F} is invariant under a specific unitary transformation by $\hat{\Sigma}_{\vec{\mu}}$ such that

$$\hat{\Sigma}_{-\vec{\mu}} \hat{F} \hat{\Sigma}_{\vec{\mu}} = \hat{F} .\tag{80}$$

Such an operator has a translationally invariant WWM symbol

$$f(\vec{V}) = f(\vec{V} + \vec{\mu})\tag{81}$$

where, since Λ is arbitrary, it is implied that $f(\vec{V})$ is independent of $\vec{V} \cdot \vec{n}_\mu$ where $\vec{n}_\mu = \vec{\mu}/|\vec{\mu}|$ is the unit vector in the $\vec{\mu}$ direction.

For $\hat{\mathcal{G}}$ describing the elements of the dynamical symmetry group, $\Delta_{CT}(\vec{V})$ corresponds to the quantum action-angle operator representation of the Wigner-Kirkwood basis. Based on the analogy in Eq. (70) and the generalized Wigner function defined in Eq. (8) we now define a *generalized action-angle* Wigner function of an arbitrary quantum state $|\psi\rangle$ as

$$W_\psi(\vec{V}) = \langle \psi | \Delta_{CT}(\vec{V}) | \psi \rangle \quad (82)$$

which can be expressed in the continuous *action eigenbasis* as

$$W_\psi(\vec{V}) = \int_{-\infty}^{\infty} \frac{d\tau_1}{2\pi} e^{i\tau_1 V_2} \langle \psi | h^\kappa(V_1 - \tau_1/2) \rangle \langle h^\kappa(V_1 + \tau_1/2) | \psi \rangle \quad (83)$$

and in the continuous *angle eigenbasis* as

$$W_\psi(\vec{V}) = \int_{-\infty}^{\infty} \frac{d\tau_2}{2\pi} e^{-i\tau_2 V_1} \langle \psi | k^\chi(V_2 - \tau_2/2) \rangle \langle k^\chi(V_2 + \tau_2/2) | \psi \rangle . \quad (84)$$

Thus, Eq's (8) and (82) are two equivalent phase space representations of the same quantum system. The former is defined in a generic canonical basis $(\hat{\mathcal{U}}, \hat{\mathcal{V}})$, whereas the latter is expressed in terms of the elements of the dynamical symmetry group of the same system. In Eq's (83) and (84) V_1 and V_2 are, by the WWM correspondence in Eq's (78) and (79), the generalized classical action-angle variables.

B. AA Wigner function for the generalized oscillator with the discrete phase space representations

In the previous sections we examined the generalized theory of quantum action-angle formalism using the continuous representations of the generalized action-angle operators. Here, we particularize this formalism to that of the representations of a generalized oscillator with a discrete spectrum in the finite -D- dimensional Hilbert space H_D by constructing the unitary canonical pair $(\hat{\mathcal{G}}, \hat{\mathcal{O}})$ as

$$\begin{pmatrix} \hat{\mathcal{G}} \\ \hat{\mathcal{O}} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \hat{\mathcal{E}}_N \\ \hat{\mathcal{E}}_\phi \end{pmatrix} \quad (85)$$

where $\hat{\mathcal{E}}_N = e^{-i\gamma_0 \hat{N}}$ with \hat{N} as defined in Eq's (A.3), (A.4) and (64). The unitary operator $\hat{\mathcal{E}}_\phi$ corresponding to the unitary angle operator $\hat{\mathcal{O}}$ above will be represented by the unitary quantum phase operator defined in Eq's (66), (67) and (68). To facilitate the correspondence with the classical case, we switch from the generalized notation $\vec{V} = (V_1, V_2)$ of the action-angle variables in section IV.A to the more standard one (J, θ) . The realization of the action angle Wigner-Kirkwood basis in the unitary number-phase basis $(\hat{\mathcal{E}}_N, \hat{\mathcal{E}}_\phi)$ has been derived in Ref. [15] for the generalized oscillator with discrete, cyclic and finite -D- dimensional Hilbert space representations as

$$\hat{\Delta}_{CT}(J, \theta) = \frac{1}{2\pi D} \sum_{\vec{m}} e^{i(\gamma_0 m_1 J - m_2 \theta)} e^{-i\gamma_0 m_1 m_2 / 2} \hat{\mathcal{E}}_N^{m_1} \hat{\mathcal{E}}_\phi^{m_2}, \quad \vec{m} \in \mathbb{Z}_D \times \mathbb{Z}_D. \quad (86)$$

In the discrete case, the set of completeness relations analogous to the continuous ones in Eq. (76) -by direct use of Eq. (86)- are

$$\begin{aligned} \hat{\Delta}_{CT}(J, \theta) &= \hat{\Delta}_{CT}^\dagger(J, \theta) \\ \int dJ \int d\theta \hat{\Delta}_{CT}(J, \theta) &= \mathbb{I} \\ \text{Tr}\{\hat{\Delta}_{CT}(J, \theta)\} &= 1/2\pi \\ \text{Tr}\{\hat{\Delta}_{CT}(J, \theta) \hat{\Delta}_{CT}(J', \theta')\} &= 1/2\pi \delta(J - J') \delta(\theta - \theta'). \end{aligned} \quad (87)$$

The AA Wigner function of $\hat{\Delta}_{CT}(J, \theta)$ in a physical state $|\psi\rangle$ is then given by¹⁵

$$W(J, \theta) = \langle \psi | \hat{\Delta}_{CT}(J, \theta) | \psi \rangle = \frac{1}{2\pi} \sum_{k=0}^{D-1} e^{-ik\theta} \langle \psi | J - k/2 \rangle \langle J + k/2 | \psi \rangle, \quad k \in \mathbb{Z} \quad (88)$$

where the states $|J \pm k/2\rangle$ are vectors in continuously shifted Fock spaces.¹⁵ Here $\{|J \pm k/2\rangle; k = \text{odd}\} \in \mathcal{F}^{(\alpha \pm 1/2)}$ and $\{|J \pm k/2\rangle; k = \text{even}\} \in \mathcal{F}^{(\alpha)}$ with α satisfying the conditions¹⁵ that $2(J - \alpha) \in \mathbb{Z}$ and $\alpha \in \mathbb{R}[0, 1)$. The definition of an arbitrary vector in the continuously shifted Fock space $\mathcal{F}^{(\beta)}$ has been given in Ref. [15] by

$$|n + \beta\rangle \equiv \frac{1}{\sqrt{D}} \sum_{\ell=0}^{D-1} e^{-i\gamma_0(n+\beta)\ell} |\phi\rangle_\ell, \quad \beta \in \mathbb{R}[0, 1), \quad |n + \beta\rangle \equiv |n + D + \beta\rangle \in \mathcal{F}^{(\beta)}. \quad (89)$$

Now, let us take $D \rightarrow \infty$ in Eq. (86) and, using Eq. (89) examine Eq. (88) for $|\psi\rangle$ being

a) a pure Fock state, i.e. $|\psi\rangle_p = |n\rangle$, n fixed

and

b) a typical mixed Fock state of the type $|\psi\rangle_m = (|n\rangle + |n-1\rangle)/\sqrt{2}$, n fixed .

a) For $|\psi\rangle_p$, and after a short calculation, Eq. (88) can be evaluated in the limit $D \rightarrow \infty$ as

$$W(J, \theta)\big|_p = \frac{1}{2\pi} \delta(n - J) \quad (90)$$

The marginal probability distributions for the J or θ variables in the pure Fock state can then be found by integrating over the other variable θ or J respectively as

$$\begin{aligned} P(J)\big|_p &\equiv \int d\theta W_p(J, \theta) = \delta(n - J) , \\ \tilde{P}(\theta)\big|_p &\equiv \int dJ W_p(J, \theta) = 1/2\pi \end{aligned} \quad (91)$$

which correctly describe the expected results for the pure Fock state.

b) The state $|\psi\rangle_m$ is the so called *split* state. For this state, using Eq. (89), we obtain

$$W(J, \theta)\big|_m = \frac{1}{4\pi} \left\{ \delta(n - J) + 2 \delta(n - J - 1/2) \cos \theta + \delta(n - J - 1) \right\} . \quad (92)$$

The marginal probability distributions yield for $|\psi\rangle_m$

$$\begin{aligned} P(J)\big|_m &= \left\{ \delta(n - J) + \delta(n - J - 1) \right\} / 2 \\ \tilde{P}(\theta)\big|_m &= (1 + \cos \theta) / 2\pi \end{aligned} \quad (93)$$

which are the correct action and angle probability distributions for the split state.

The Eq. (88) also provides the correct time dependence for the AA Wigner function in the QHO limit. In order to observe this we will start with the generalized oscillator in Eq. (63).

The time dependence of the AA Wigner function is given by the standard expression

$$i \frac{d}{dt} W_\psi(J, \theta) = \langle \psi | [\hat{\mathcal{H}}, \hat{\Delta}_{CT}(J, \theta)] | \psi \rangle \quad (94)$$

or, equivalently, in terms of the WWM symbol $h(J, \theta)$ of $\hat{\mathcal{H}}$ as³⁰

$$\frac{d}{dt} W_\psi(J, \theta) = \left\{ h(J, \theta) * W_\psi(J, \theta) - W_\psi(J, \theta) * h(J, \theta) \right\} , \quad * = \exp \left\{ \frac{i}{2} \left[\overleftarrow{\frac{\partial}{\partial J}} \overrightarrow{\frac{\partial}{\partial \theta}} - \overleftarrow{\frac{\partial}{\partial \theta}} \overrightarrow{\frac{\partial}{\partial J}} \right] \right\} \quad (95)$$

where $(h * W_\psi - W_\psi * h) \equiv \{h, W_\psi\}_{MB}$ is the Moyal (sine) bracket.³⁰ The calculation of Eq. (95) requires the knowledge of $h(J, \theta)$. This can be obtained by using the completeness equations in (87) and the Hamiltonian operator $\hat{\mathcal{H}}$ in (63) as

$$\hat{\mathcal{H}} = \int dJ \int d\theta h(J, \theta) \hat{\Delta}_{CT}(J, \theta), \quad \text{where} \quad h(J, \theta) = \text{Tr}\{\hat{\mathcal{H}}\hat{\Delta}_{CT}(J, \theta)\}. \quad (96)$$

The trace operation can be conveniently carried in the discrete finite dimensional cyclic eigenspace $\{|n\rangle\} = \{|n\rangle_{0 \leq n \leq (D-1)}; |n+D\rangle \equiv |n\rangle\}$ of the Hamiltonian $\hat{\mathcal{H}}$. Since for the diagonal matrix elements $\langle n|\hat{\Delta}_{CT}(J, \theta)|n\rangle = W(J, \theta)|_p$ are Wigner functions of the pure Fock states, we can also directly use the expression (90) in the calculation of the trace. We find that

$$h(J, \theta) = \mathcal{H}(J) \quad (97)$$

Using Eq. (97) in Eq. (95)

$$\frac{d}{dt}W_\psi(J, \theta) = -i \left\{ \mathcal{H}\left(J + \frac{i}{2} \frac{\partial}{\partial \theta}\right) - \mathcal{H}\left(J - \frac{i}{2} \frac{\partial}{\partial \theta}\right) \right\} W_\psi(J, \theta) \quad (98)$$

Eq. (98) is the equation of motion of the action-angle Wigner function for an arbitrary generalized cyclic oscillator in Eq. (63). Now, we apply Eq. (98) to the QHO case where the Hamiltonian in Eq. (63) is a linear operator of \hat{N} , let us say $\mathcal{H}(\hat{N}) = \omega \hat{N}$ with ω describing the oscillator frequency. Then by Eq. (96) and (97), $h(J, \theta) = \omega J$. For the QHO Eq. (98) then yields,

$$\left\{ \frac{d}{dt} - \omega \frac{\partial}{\partial \theta} \right\} W_\psi(J, \theta) = 0, \quad \implies \quad \theta = \theta(t) = \omega t \quad (99)$$

namely, the time evolution of the QHO action-angle Wigner function in the phase space takes place on the classical manifold ($J = \text{constant}, \theta = \omega t$) as expected. By Eq. (90), The AA Wigner function for the pure Fock state is static. The full time dependent solution of the AA Wigner function, for instance, for the split state and the corresponding marginal probability distributions can be completely determined by inserting the solution of $\theta(t)$ in Eq. (99) into Eq. (92) and (93).

V. CONCLUSIONS

The canonical-algebraic connection between the quantum phase problem and the quantum phase space has already been noticed by some other workers recently. In particular, using the generators of the angular momentum $su(2)$ algebra and its dual in terms of the Hermitian canonical phase operators Vourdas has studied³¹ an equivalent canonical pair to $(\hat{\mathcal{G}}, \hat{\mathcal{O}})$ as defined in this work. Our specific aim in this publication was to further the canonical algebraic approach introduced in Ref. [15] to unify the formulation of quantum phase with that of the algebraic theory of quantum canonical transformations. In this context, we investigated the generators of quantum canonical transformations, their unitary canonical partners in the Schwinger sense, as well as their action on the functions of canonical variables of the quantum phase space, in particular the Wigner function. Through this connection, the quantum phase is formally established as the unitary canonical partner of the quantum action operator which is also demonstrated for the one dimensional generalized oscillator.

VI. ACKNOWLEDGMENTS

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APPENDIX

In Ref. [15] we have examined two subalgebraic realizations of the discrete-cyclic Schwinger operator basis $\hat{\mathcal{S}}_{\vec{m}}$. In the following we will have a brief summary of them. Based on a fixed pair of vectors \vec{m}, \vec{m}' the sine algebra generated by $\hat{\mathcal{S}}_{\vec{m}}$ supports two sub algebraic realizations.¹⁵ The standard $u_q sl(2)$ sub algebraic realization is obtained by constructing the generators

$$\begin{aligned}\hat{J}_- &\equiv d \hat{\mathcal{S}}_{\vec{m}} + d' \hat{\mathcal{S}}_{\vec{m}'} \\ \hat{J}_+ &\equiv d^* \hat{\mathcal{S}}_{-\vec{m}} + d'^* \hat{\mathcal{S}}_{-\vec{m}'} \\ \hat{L} &\equiv \hat{\mathcal{S}}_{\vec{m}-\vec{m}'} = q^{\hat{J}_3 + \frac{D}{2}}\end{aligned}\tag{A.1}$$

where $d d'^* = d^* d' = -(q^{1/2} - q^{-1/2})^{-2}$ and $q = e^{-i\gamma_0 \vec{m} \times \vec{m}'}$ so that

$$\hat{J}_{\mp} \hat{L} = q^{\pm 1} \hat{L} \hat{J}_{\mp}, \quad [\hat{J}_-, \hat{J}_+] = -\frac{\hat{L} - \hat{L}^{-1}}{q^{1/2} - q^{-1/2}} = -[\hat{J}_3 + \frac{D}{2}].\tag{A.2}$$

This particular sub algebraic realization is sometimes referred as the magnetic translation group.³²

On the other hand, more importantly for the purpose of this work, a second class of sub algebraic realizations exist in the form of an *admissible* q-oscillator algebra which can be obtained by defining

$$\begin{aligned}\hat{A} &\equiv d \hat{\mathcal{S}}_{\vec{m}} + d' \hat{\mathcal{S}}_{\vec{m}'} \\ \hat{A}^\dagger &\equiv d^* \hat{\mathcal{S}}_{-\vec{m}} + d'^* \hat{\mathcal{S}}_{-\vec{m}'} \\ \hat{Q} &\equiv q^{-\hat{N} - (D-1)/2} = q^{1/2} \hat{\mathcal{S}}_{-(\vec{m}-\vec{m}')} \end{aligned}\tag{A.3}$$

where $d d'^* = -d^* d' = -(q - q^{-1})^{-1}$ and $q = e^{-i\gamma_0 \vec{m} \times \vec{m}'}$ so that

$$\begin{aligned}\hat{A} \hat{Q} &= q^{-1} \hat{Q} \hat{A}, \quad \hat{A}^\dagger \hat{Q} = q \hat{Q} \hat{A}^\dagger, \\ \hat{A}^\dagger \hat{A} &= C + [\hat{N}], \quad \text{such that} \\ \hat{A} \hat{A}^\dagger - q \hat{A}^\dagger \hat{A} &= (1 - q) C + \hat{Q}\end{aligned}\tag{A.4}$$

where $[\hat{N}] = (\hat{Q}^{-1} - \hat{Q})/(q - q^{-1})$ and $C = (|\sin(\gamma_0 \vec{m} \times \vec{m}')|)^{-1}$. Eq's (A.4) imply that the q-oscillator spectrum is non-negative (i.e. $0 \leq \|\hat{A}^\dagger \hat{A}\|$ where the spectrum is given by the eigenvalues of the operator $\hat{A}^\dagger \hat{A} = C + [\hat{N}]$) which, further implies that the Hilbert space is

spanned by vectors with admissible (non-negative) norm. It was shown in Ref. [15] that the admissible q -oscillator algebra in Eq. (A.3) and (A.4) is crucial in establishing a canonical-algebraic approach to the quantum phase problem. Interested reader can find more detailed discussions of the admissible q -oscillator realizations therein.

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